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Computing the Topological Entropy of Maps of the Interval with Three Monotone Pieces

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An algorithm is presented for computing the topological entropy of a piecewise monotone map of the interval having three monotone pieces. The accuracy of the algorithm is discussed and some graphs of the topological entropy obtained using the algorithm are displayed. Some of the ideas behind the algorithm have application to piecewise monotone functions with more than three monotone pieces.

KEY WORDS: Topological entropy; piecewise monotone map; chaotic dynamical system; kneading matrix.

1. BACKGROUND

The topological entropy of a map gives a quantitative measure of the complexity of a system modeled by iterating the map. Collet *et al.*[4] and Block *et al.*[2] have given algorithms to compute the topological entropy in the special case of unimodal maps of an interval. An algorithm to compute the entropy in the more general case of piecewise monotone maps of an interval was recently given by Góra and Boyarsky.[6] Although more general, their algorithm was not as efficient nor as accurate as that in ref. 2 for unimodal maps. Although refs. 2 and 4 both use kneading sequences and are based on the work of Milnor and Thurston,[8] the approaches are quite different. In ref. 4, use is made of an analytic function whose coefficients are determined by the kneading sequence and one of whose roots can be used to determine the topological entropy. In ref. 2, the algorithm is based on the fact that the tent maps have known topological entropy and the kneading sequences of these maps can be used to determine if the

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topological entropy of one of these maps is greater than or less than that of another unimodal map.

This paper continues the approach of ref. 2 and extends it to mappings with three monotone pieces. The method is based on itineraries and kneading matrices. Let $h(f)$ denote the topological entropy of the map f. We associate a kneading matrix $K(f)$ to each piecewise monotone map f and define a partial ordering on kneading matrices so that if $K(f) \leqslant K(g)$, then $h(f) \leqslant h(g)$. This result is obtained by combining a result of Misiurewicz and Szlenk[7] with a modification of a result of Baldwin.[1]

Using this result, we are able to obtain an effective algorithm for computing the topological entropy for maps with three monotone pieces. The algorithm is based on comparing the kneading matrix of the map whose entropy we wish to compute with the kneading matrix of a map whose entropy is known. It is in this respect that this paper is a natural extension of ref. 2.

The main obstacle in implementing an algorithm based on these ideas is that for particular maps f and g it may happen that neither $K(f) \leqslant K(g)$ nor $K(f) \geqslant K(g)$. In Section 4 we consider comparisons between an arbitrary map g of the interval with three monotone pieces and a map f_s with slope a constant absolute value s with three monotone pieces. In that section we show that except for a countable number of values of s, there is always some map f_s for which $K(g) \leqslant K(f_s)$ or $K(g) \geqslant K(f_s)$. In Section 5 a practical algorithm is presented. The algorithm appears to be more accurate and require less computer time than the algorithm given in ref. 6 for three monotone pieces. The algorithm presented in this paper has two advantages. An upper bound on the number of function evaluations can be prescribed and the error of the estimate of the topological entropy can also be prescribed. In Section 6 some pragmatic items for implementing the algorithm are dealt with. A numerical experiment is presented which gives strong empirical evidence that roundoff error and other possible obstacles discussed in the paper do not interfere with the practical implementation of the algorithm. Section 7 presents some graphs of the topological entropy obtained using our algorithm.

Before giving a detailed presentation of our algorithm and its theoretical justification, we discuss heuristically the problem of computing the topological entropy and the role that we imagine our work may play in a more general algorithm.

2. HEURISTIC DISCUSSION

The main tool of this paper is Theorem 3.3: if $K(f) \leqslant K(g)$, then $h(f) \leqslant h(g)$. The idea behind this result is that if $K(f) \leqslant K(g)$, then for each n there are as many distinct itineraries for g of length n as there are for f. The growth rate of the number of itineraries of length n determines the topological entropy. This implies that $h(f) \leqslant h(g)$. There is no requirement in that theorem that f or g have three monotone pieces, merely that they have the same number of monotone pieces. Whenever f is piecewise linear with slope $\pm s$ on each monotone piece and $K(f) \leqslant K(g)$, then since $h(f) = \log(s)$, $\log(s) \leqslant h(g)$. Similarly, if $K(g) \leqslant K(f)$, then $h(g) \leqslant \log(s)$.

In Section 4 we show that if g has three monotone pieces, then for all but countably many $s \in [0, \log(3)]$ there is an f with three monotone pieces and with slope $\pm s$ on each monotone piece such that $K(f) \leqslant K(g)$ or $K(g) \leqslant K(f)$. To obtain this comparison, we are prompted to consider the family $PL(s)$ of piecewise monotone functions with three monotone pieces having slope $\pm s$ on each monotone piece. What allows us to find the required comparison is the fact that this is a one-parameter family of functions.

Now, for a function g with more than three monotone pieces, one needs a way of guaranteeing that there is an f with the same number of monotone pieces and with slope $\pm s$ on each monotone piece such that $K(f) \leqslant K(g)$ or $K(g) \leqslant K(f)$. What makes the search for this required comparison more complicated when g has more than three monotone pieces is that the corresponding family $PL(s)$ will no longer be a one-parameter family. One would need an algorithm to search through this family of such functions f to find the appropriate comparison. One would also need some theorem which would guarantee that such a comparison would always exist. We feel that the comparisons will always exist and that an effective algorithm for searching for this comparison will be found. The advantage to this approach is that it can be used to determine the topological entropy to any precision desired.

One should expect that as the number of monotone pieces increases, the computation necessary to compute the topological entropy will also increase. It is useful to compare the amount of computation required in the algorithm for unimodal maps studied in ref. 2 with that presented here for functions with three monotone pieces. Consider a unimodal function $f: [0, 1] \to [0, 1]$. If we assume that 50 terms were used in the kneading sequence and that computations were carried to ten-digit accuracy, then at most 2000 function evaluations are required in the algorithm in ref. 2 to determine the topological entropy. If we use 50 terms in the kneading matrix in the algorithm described in this paper, then at most 160,000 function evaluations will be necessary. This is still a reasonable number, especially since with this protocol one usually obtains the topological entropy to several digits accuracy. This number of computations usually takes only a few seconds. What would we expect to happen as the number of monotone pieces is increased further? It is likely that the number of function evaluations that will be required will be at most $(n-1) \times 50 \times 40^{n-1}$, where n is the number of monotone pieces of the function. Thus, although this approach promises precise estimates of the topological entropy with modest computational expense for functions with a small number of monotone pieces, the computation required as n grows will quickly get out of hand. Additional insight will be needed to reduce the number of computations to a reasonable level for large n.

Some remarks are in order concerning wider uses of the kneading matrix. The basic features of the kneading matrix are given in ref. 8. The kneading matrix does not completely determine the topological conjugacy class of a map of the interval. However, it does play a significant role in that determination. For a complete topological classification of the piecewise monotone maps of the interval, see the paper by Baldwin.[1] We are indebted to that paper for the ideas leading to Theorem 3.3.

3. RESULTS FOR ARBITRARY PIECEWISE MONOTONE MAPS

Let f be a function defined on an interval $[a, b]$. We say that f is *piecewise monotone* if f is continuous and there are points

$$a = z_0 < z_1 < \cdots < z_k = b$$

such that for each $i = 0,..., k-1$, f is either strictly increasing or strictly decreasing on $[z_i, z_{i+1}]$.

If f is piecewise monotone on $[a, b]$, let $N = N(f)$ denote the number of maximal monotone intervals for f, and let

$$a = t_0(f) < t_2(f) < t_4(f) \cdots < t_{2N}(f) = b$$

denote the endpoints of these maximal monotone intervals. Note that these points are subscripted by even integers. We are reserving the missing odd integers to subscript the intervals between these points. Note that each $t_{2i}(f)$ is a relative maximum or a relative minimum for f. We write t_{2i} in place of $t_{2i}(f)$ if the map f is understood.

We define a collection of intervals and points $J_k = J_k(f)$ for $k = 0,..., 2N$ by setting $J_k = \{t_k\}$ if k is even and $J_k = (t_{k-1}, t_{k+1})$ if k is odd. See Fig. 1 for a diagram. For each $x \in [a, b]$ the itinerary of x with respect to f is the infinite sequence of integers

$$I_f(x) = (a_0(x), a_1(x),...)$$

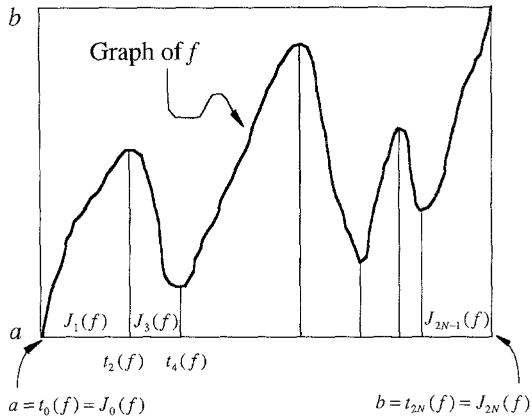

Fig. 1. A typical piecewise monotone function f. Here $N = 7$.

where $a_j(x) = k$ if and only if $f^j(x) \in J_k$. Here, f^j denotes composition of f with itself j times (and f^0 denotes the identity map). The itinerary is a way of coding the successive iterates of a point. An odd integer in the itinerary indicates that the corresponding iterate is in one of the open maximal monotone intervals, while an even integer indicates that the corresponding iterate is exactly a relative extreme point. If $I_f(x) = (a_0(x), a_1(x),...)$ is any itinerary and i is a positive integer, then the finite sequence

$$(a_0(x), a_1(x),..., a_i(x))$$

is called a *finite itinerary* of length $i+1$.

We define the *sign function* for f to be the function with domain $\{1, 3, 5,..., 2N-1\}$ defined by $\text{sign}(k) = 1$ if f is increasing on J_k and $\text{sign}(k) = -1$ if f is decreasing on J_k. Note that the values of the sign function alternate between $+1$ and -1.

Let f and g be piecewise monotone functions with the same number of maximal monotone intervals and the same sign function. Let

$$\mathbf{a} = (a_0(x), a_1(x),...) = I_f(x)$$

for some x in the domain of f, and let

$$\mathbf{b} = (b_0(y), b_1(y),...) = I_g(y)$$

for some y in the domain of g. We say $\mathbf{a} < \mathbf{b}$ if both of the following hold:

1. $a_i \neq b_i$ for at least one i.

2. Let k be the smallest nonnegative integer with $a_k \neq b_k$. If $k=0$, then $a_0 < b_0$, and if $k>0$, then $a_0,..., a_{k-1}$ are all odd and

$$\left(\prod_{i=0}^{k-1} \operatorname{sign}(a_i)\right) a_k < \left(\prod_{i=0}^{k-1} \operatorname{sign}(b_i)\right) b_k$$

Note that in the above inequality,

$$\prod_{i=0}^{k-1} \operatorname{sign}(a_i) = \prod_{i=0}^{k-1} \operatorname{sign}(b_i)$$

This product will be -1 if the itinerary lies in an interval on which the map is decreasing an odd number of times and will be $+1$ otherwise. As usual, we will write $\mathbf{a} \leqslant \mathbf{b}$ if either $\mathbf{a} < \mathbf{b}$ or $\mathbf{a} = \mathbf{b}$, $\mathbf{a} > \mathbf{b}$ if $\mathbf{b} < \mathbf{a}$, and $\mathbf{a} \geqslant \mathbf{b}$ if $\mathbf{b} \leqslant \mathbf{a}$.

The *kneading matrix* of f is the ordered n-tuple

$$K(f) = (I_f(f(t_0(f))),..., I_f(f(t_{2N}(f))))$$

Thus, the kneading matrix encodes the itineraries of the images of the relative extreme points. Although this definition is formally different than that given in,[8] it carries the same essential information and is more practical for our purposes.

We say $K(f) \leqslant K(g)$ if $I_f(f(t_i(f))) \leqslant I_g(g(t_i(g)))$ for each i such that $t_i(f)$ and $t_i(g)$ are relative maxima and $I_f(f(t_i(f))) \geqslant I_g(g(t_i(g)))$ for each i such that $t_i(f)$ and $t_i(g)$ are relative minima. Note that it may happen that neither $K(f) \leqslant K(g)$ or $K(g) \leqslant K(f)$.

Our first lemma may be obtained by a slight modification of the proof of Lemma 8 of ref. 1.

Lemma 3.1. Suppose that $K(f) \leqslant K(g)$. Let $\mathbf{a}$ be a finite itinerary of some point under f with every element of $\mathbf{a}$ odd. Then there is a point in the domain of g with finite itinerary $\mathbf{a}$.

This lemma is useful, since the entropy of a piecewise monotone map may be determined by the finite itineraries. More precisely, we have the following lemma, which follows from Lemma 6 of ref. 7.

Lemma 3.2. Let f be piecewise monotone. For each positive integer k, let $O_k(f)$ denote the number of distinct finite itineraries of length k (of points under f) with each element in the finite itinerary odd. Then

$$h(f) = \lim_{n \to \infty} \frac{1}{n} \log(O_n(f))$$

where $h(f)$ is the topological entropy of the map f.

It follows from Lemma 3.1 that if $K(f) \leqslant K(g)$, then $O_k(f) \leqslant O_k(g)$ for each positive integer k. Combining this fact with Lemma 3.2, we immediately obtain the following.

Theorem 3.3. If $K(f) \leqslant K(g)$, then $h(f) \leqslant h(g)$.

This theorem will be the theoretical basis for the algorithm we develop. Suppose that f is a piecewise monotone map on I with three monotone pieces. The next section shows that for all $s \in (0, 3) \backslash A$ where A is a countable set, there is a g_s having three monotone pieces with slope $\pm s$ on each monotone piece such that $K(f) \leqslant K(g_s)$ or $K(f) \geqslant K(g_s)$.

It also should be pointed out that there is no loss of generality in assuming that the map $f: I \to I$ takes the endpoints of I to endpoints. The reason for this is that we can replace the map f by another map g having the same number of monotone pieces and having the same topological entropy with g taking the endpoints of I to endpoints. If this is not clear, then simply observe that, given any map $f: I \to I$, we can construct a map $g: I' \to I'$, where $I' = [a', b']$, with the following properties. (1) The interval I will be contained in I'. (2) The restriction of g to I will be f, $g|I \equiv f$. (3) The map g will have $g(a') \in \{a', b'\}$ and $g(b') \in \{a', b'\}$. (4) The map g will be linear on $[a', a]$ and on $[b, b']$. (5) The map g will have the same number of monotone pieces as f. Then g will have the same topological entropy as f. Thus, we can substitute g and I' for f and I in the algorithm and g will take the endpoints of I' to endpoints of I'. All of our examples are on the unit interval $I = [0, 1]$ and satisfy $f(0) = 0$ and $f(1) = 1$.

4. MAPS WITH THREE MONOTONE PIECES

Given a number s, with $1 < s < 3$, we consider the family $PL(s)$ of all piecewise linear maps f from $[0, 1]$ to itself satisfying the following three conditions.

1. $f(0) = 0$ and $f(1) = 1$.
2. f has exactly three linear pieces.
3. f has slope s on the first and third linear piece and slope $-s$ on the second linear piece.

It follows from the third condition that if $f \in PL(s)$, then $h(f) = \log(s)$.

Note that for each $f \in PL(s)$, $t_0(f) = 0$, $t_6(f) = 1$, and $t_4(f)$ is determined by $t_2(f)$. Explicitly,

$$(t_4(f), f(t_4(f))) = \left(t_2(f) + \frac{s-1}{2s}, st_2(f) - \frac{s-1}{2}\right)$$

This is illustrated in Fig. 2. Thus, for each s, $PL(s)$ consists of a one-parameter family of maps f_t where we let $t = t_2(f)$, and (as can be easily computed) $(s-1)/2s \leqslant t \leqslant 1/s$.

Our goal is to prove that for all but countably many s with $1 < s < 3$ the following holds. Given a map g with three monotone pieces such that $g(0) = 0$ and $g(1) = 1$, there is a map $f_t \in PL(s)$ such that either $K(g) \leqslant K(f_t)$ or $K(g) \geqslant K(f_t)$. This will enable us to apply Theorem 1.3. To prove this, we need several preliminary results.

Let $\mathfrak{I}_3$ denote the set of all possible itineraries of points under maps with exactly three monotone pieces with the same sign function as the maps in $PL(s)$.

Lemma 4.1. Let $I \in \mathfrak{I}_3$ be a fixed itinerary. Suppose $f_t \in PL(s)$, and suppose one of the following holds with $j = 2$ or 4:

$$I_{f_t}(f_t(t_j(f_t))) > I \tag{1}$$

$$I_{f_t}(f_t(t_j(f_t))) < I \tag{2}$$

Furthermore, suppose that $I_{f_t}(f_t(t_j(f_t)))$ consists of only odd numbers. Then the corresponding inequality holds for all $f_{t'}$ in $PL(s)$ with t' sufficiently close to t.

Proof. Let

$$I_{f_t}(f_t(t_j(f_t))) = \mathbf{a} = (a_0, a_1, a_2, \ldots)$$

and let $I = \mathbf{b} = (b_0, b_1, b_2, \ldots)$. Let k denote the smallest nonnegative integer with $a_k \neq b_k$. The inequality (1) or (2) is completely determined by $(a_0, a_1, a_2, \ldots, a_k)$ and $(b_0, b_1, b_2, \ldots, b_k)$.

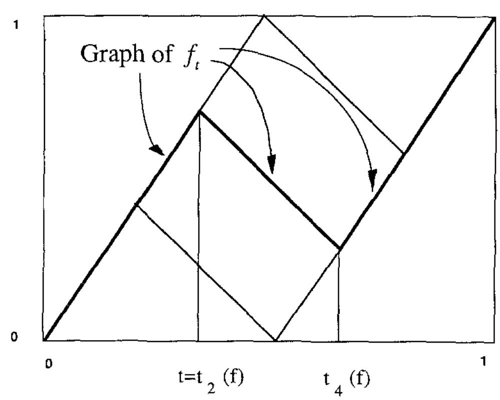

Fig. 2. The family of piecewise linear maps of the interval with slope $\pm s$ with three monotone pieces. Both the x and y axes are [0, 1].

For t' sufficiently close to t, each of the two turning points of $f_{t'}$ as well as the first k iterates under $f_{t'}$ will be close to the corresponding turning points and iterates under f_t. Also, the intervals used to compute itineraries under $f_{t'}$ will be close to the corresponding intervals for f_t. Since $a_0, a_1, ..., a_k$ are all odd, it follows that for t' sufficiently close to t, if

$$I_{f_{t'}}(f_{t'}(t_j(f_{t'}))) = \mathbf{a}' = (a'_0, a'_1, a'_2, ...)$$

then $(a'_0, a'_1, a'_2, ..., a'_k) = (a_0, a_1, a_2, ..., a_k)$. The conclusion of the lemma then follows. ∎

A piecewise linear map f of an interval to itself is said to be a *linear Markov map* if each endpoint of a linear piece has a finite orbit.

We remark that for certain values of s, the family $PL(s)$ contains linear Markov maps. For example, if $s = (1 + \sqrt{5})/2$ and

$$t = t_2(f_t) = \frac{\sqrt{5} - 1}{4}$$

then the map $f_t \in PL(s)$ is linear Markov. For this map f_t, $f_t(t_2(f_t)) = t_4(f_t) = 1/2$, $f_t(1/2) = (3 - \sqrt{5})/4$, and $f_t((3 - \sqrt{5})/4) = (\sqrt{5} - 1)/4 = t_2(f_t)$. Thus, $t_2(f_t)$ and $t_4(f_t)$ are in the same orbit of period 3.

Proposition 4.2. Let g be a map with three monotone pieces such that $g(0) = 0$, $g(1) = 1$, and g has the same sign function as the maps in $PL(s)$. Suppose that for some fixed value of s, $PL(s)$ does not contain a linear Markov map. Then there is a map $f_t \in PL(s)$ such that $K(g) \leqslant K(f_t)$ or $K(g) \geqslant K(f_t)$.

Proof. First, suppose that for some $f_t \in PL(s)$ we have

$$I_{f_t}(f_t(t_2(f_t))) = I_g(g(t_2(g))) \tag{3}$$

If $I_{f_t}(f_t(t_2(f_t)))$ contains an even number, then since f_t is not Markov, $\mathbf{a} = I_{f_t}(f_t(t_4(f_t)))$ does not contain an even number. Thus, in this case, if $\mathbf{b} = I_g(g(t_4(g)))$, we must have one of $\mathbf{a} = \mathbf{b}$, $\mathbf{a} < \mathbf{b}$, or $\mathbf{b} < \mathbf{a}$, and hence either $K(g) \leqslant K(f_t)$ or $K(g) \geqslant K(f_t)$. If $I_{f_t}(f_t(t_2(f_t)))$ does not contain an even number, then $\mathbf{a}$ may contain an even number and we may have neither $\mathbf{a} < \mathbf{b}$ nor $\mathbf{b} < \mathbf{a}$. However, in this case if $\mathbf{a} = (a_0, a_1, ...)$ and $\mathbf{b} = (b_0, b_1, ...)$, then for some k we have $a_i = b_i$ for $0 \leqslant i \leqslant k$ and $a_k = b_k$ is even. Since we already know that the itineraries of the other three critical points of f_t [i.e., the critical points 0, 1, and $t_2(f_t)$] agree with the itineraries of the corresponding critical points of g, it follows that $\mathbf{a} = \mathbf{b}$. Hence, $K(g) = K(f_t)$. Thus, the conclusion of the proposition follows whenever the equality (3) holds. Similarly, the conclusion of the proposition follows whenever

$$I_{f_t}(f_t(t_4(f_t))) = I_g(g(t_4(g))) \tag{4}$$

Thus, we may assume that neither (3) nor (4) holds. Let c denote the infimum of the set of $t \in [(s-1)/2s, 1/s]$ such that

$$I_{f_t}(f_t(t_2(f_t))) > I_g(g(t_2(g)))$$

Note that this set is nonempty since the inequality holds when $t = 1/s$. We divide the proof into three cases.

Case 1. $(s-1)/2s < c < 1/s$.
For all $t \in [(s-1)/s, c)$ we have

$$I_{f_t}(f_t(t_2(f_t))) < I_g(g(t_2(g)))$$

On the other hand, for any $\varepsilon > 0$, the interval $[c, c+\varepsilon)$ contains a value of t with

$$I_{f_t}(f_t(t_2(f_t))) > I_g(g(t_2(g)))$$

It follows from Lemma 4.1 that $I_{f_c}(f_c(t_2(f_c)))$ contains an even number. Since f_c is not a linear Markov map, it follows that $I_{f_c}(f_c(t_4(f_c)))$ does not contain an even number. Hence, by Lemma 4.1, either for all t in some open interval containing c,

$$I_{f_t}(f_t(t_4(f_t))) > I_g(g(t_4(g)))$$

or for all t in some open interval containing c,

$$I_{f_t}(f_t(t_4(f_t))) < I_g(g(t_4(g)))$$

In either case, the conclusion follows.

Case 2. $c = (s-1)/2s$.
Since $f_c(t_4(f_c)) = 0$, and f_c is not a linear Markov map, it follows that $I_{f_c}(f_c(t_2(f_c)))$ does not contain an even number. Hence, by Lemma 2.1 and the choice of c, we must have

$$I_{f_c}(f_c(t_2(f_c))) > I_g(g(t_2(g)))$$

and hence $K(f_c) \geqslant K(g)$.

Case 3. $c = 1/s$.
In this case,

$$I_{f_c}(f_c(t_2(f_c))) > I_g(g(t_2(g)))$$

but $I_{f_t}(f_t(t_2(f_t))) < I_g(g(t_2(g)))$ for all $t < c$. As in Case 1, the conclusion follows by using Lemma 4.1. ∎

Given a linear Markov map f, we associate a matrix $A = (a_{ij})$ to f as follows. Let $I_1, \ldots, I_n$ denote the intervals joining adjacent points in the set of points in the orbit of an endpoint of a linear piece. Assign a_{ij} as follows:

$$a_{ij} = \begin{cases} 1 & \text{if } f(I_i) \supset I_j \\ 0 & \text{otherwise} \end{cases}$$

Since A has entries 0 and 1, there is a unique, real, nonnegative eigenvalue λ of A satisfying $\lambda \geqslant |\mu|$ for all eigenvalues μ of A. We call λ the *maximal eigenvalue* of A.

The following proposition is well known.

Proposition 4.3. Let f be a linear Markov map. Let λ be the maximal eigenvalue of the matrix associated to f. Then the following holds:

$$h(f) = \begin{cases} \log \lambda & \text{if } \lambda > 0 \\ 0 & \text{if } \lambda = 0 \end{cases}$$

Theorem 4.4. For all but countably many s with $1 < s < 3$, the following holds. Given an arbitrary map g with $g(0) = 0$, $g(1) = 1$, such that g has exactly three monotone pieces, there is a map $f_t \in PL(s)$ such that either $K(g) \leqslant K(f_t)$ or $K(g) \geqslant K(f_t)$.

Proof. Suppose that for a given s the conclusion does not hold. Then by Proposition 4.2, $PL(s)$ contains a linear Markov map f. Since $h(f) = \log(s)$, it follows from Proposition 4.3 that s is the maximal eigenvalue of a matrix with entries zero and one. Since there are only countably many such matrices, there are at most countably many choices for s. This proves the theorem. ∎

5. A THEORETICAL ALGORITHM

In this section we present an algorithm to compute the topological entropy of a map of the interval with three monotone pieces. Without loss of generality we assume that $g: I \to I$ with $I = [0, 1]$. Furthermore, as pointed out in Section 1, we may assume that $g(0) \in \{0, 1\}$ and $g(1) \in \{0, 1\}$. We assume that $g(0) = 0$ and that $g(1) = 1$. The case that $g(0) = 1$ and $g(1) = 0$ can be handled in a similar fashion.

We know that the topological entropy of f with three monotone pieces must lie in the interval $I_1 = [0, \log(3)]$. The basic idea of the algorithm is to approximate $h(f)$ by dividing I_1 into two intervals with $\log(2)$ as the dividing point and determining which one contains $h(g)$. Call that interval I_2. Determining which interval contains $h(g)$ is accomplished by finding a comparison of g with an $f_s \in PL(s)$, $s=2$. The comparison will determine whether g has topological entropy greater than or less than log(2). Having determined $I_2=[a, b]$, we repeat the process on I_2 to find a smaller interval I_3 containing $h(g)$. The interval I_3 is determined by finding a comparison of g with an $f_s \in PL(s)$, $s=(e^a+e^b)/2$. The procedure should be fairly clear at this point.

The difficult part in developing an algorithm using the above philosophy is in finding a map $f_s \in PL(s)$, $s=(e^a+e^b)/2$ such that f_s has a comparison with g.

We let $K_N(g)$ be the first N terms in the kneading matrix of g. We let $I_{g,N}(g(t_i(g)))$ denote the first N terms in the itinerary of $g(t_i(g))$ under g, and let $I_{f,N}(f_s(t_i(f_s)))$ denote the first N terms in the itinerary of $f_s(t_i(f_s))$ under f_s.

The Algorithm

First choose a number N which will be the number of terms in the kneading matrix used for comparison. In our previous paper the number N could be determined beforehand so that the estimate of the topological entropy $h(g)$ would be accurate to within a prescribed $\varepsilon>0$. In Section 4 we determine empirically the number of digits of accuracy that can be expected using different numbers of terms in the kneading matrix.

Assume that $g: I \to I$ is a piecewise monotone map of the unit interval with three monotone pieces satisfying $g(0)=0$ and $g(1)=1$. Let $\varepsilon>0$ be given. Let k be the number of digits of accuracy used in the computations.

Step 1. Compute $I_{g,N}(g(t_2(g)))$ and $I_{g,N}(g(t_4(g)))$.

Step 2. Let $I=[1, 3]=[a, b]$.

Step 3. Let $s=(a+b)/2$.

Step 4. Let $c=(s-1)/2s$ and $d=1/s$.

Step 5. Take the $f_s \in PL(s)$ with $t_2(f_s)=c=(s-1)/2s$. Note that s and $t_2(f_s)$ determine the value of $t_4(f_s)$.

Step 6. Compute $I_{f,N}(f_s(t_2(f_s)))$ and $I_{f,N}(f_s(t_4(f_s)))$.

Step 7. Compare $I_{f,N}(f_s(t_2(f_s)))$ with $I_{g,N}(g(t_2(g)))$ and $I_{f,N}(f_s(t_4(f_s)))$ with $I_{g,N}(g(t_4(g)))$. If $I_{f,N}(f_s(t_2(f_s)))<I_{g,N}(g(t_2(g)))$ and $I_{f,N}(f_s(t_4(f_s)))>I_{g,N}(g(t_4(g)))$, then let $a=s$ and go to Step 16. If $I_{f,N}(f_s(t_2(f_s)))>I_{g,N}(g(t_2(g)))$ and $I_{f,N}(f_s(t_4(f_s)))<I_{g,N}(g(t_4(g)))$, then let $b=s$ and go to Step 16. Otherwise continue to the next step.

Step 8. Take the $f_s \in PL(s)$ with $t_2(f_s) = d = 1/s$. Note that s and $t_2(f_s)$ determines the value of $t_4(f_s)$.

Step 9. Compute $I_{f,N}(f_s(t_2(f_s)))$ and $I_{f,N}(f_s(t_4(f_s)))$.

Step 10. Compare $I_{f,N}(f_s(t_2(f_s)))$ with $I_{g,N}(g(t_2(g)))$ and $I_{f,N}(f_s(t_4(f_s)))$ with $I_{g,N}(g(t_4(g)))$. If $I_{f,N}(f_s(t_2(f_s))) < I_{g,N}(g(t_2(g)))$ and $I_{f,N}(f_s(t_4(f_s))) > I_{g,N}(g(t_4(g)))$, then let $a = s$ and go to Step 16. If $I_{f,N}(f_s(t_2(f_s))) > I_{g,N}(g(t_2(g)))$ and $I_{f,N}(f_s(t_4(f_s))) < I_{g,N}(g(t_4(g)))$, then let $b = s$ and go to Step 16. Otherwise continue to the next step.

Step 11. If $d - c < 10^{-k}$, go to Step 17. Otherwise continue to the next step.

Step 12. Let $f_s \in PL(s)$ have slope s given in Step 2 and let $t_2(f_s) = (c + d)/2$.

Step 13. Compute $I_{f,N}(f_s(t_2(f_s)))$ and $I_{f,N}(f_s(t_4(f_s)))$.

Step 14. Compare $I_{f,N}(f_s(t_2(f_s)))$ with $I_{g,N}(g(t_2(g)))$ and $I_{f,N}(f_s(t_4(f_s)))$ with $I_{g,N}(g(t_4(g)))$. If $I_{f,N}(f_s(t_2(f_s))) < I_{g,N}(g(t_2(g)))$ and $I_{f,N}(f_s(t_4(f_s))) > I_{g,N}(g(t_4(g)))$, then let $a = s$ and go to Step 16. If $I_{f,N}(f_s(t_2(f_s))) > I_{g,N}(g(t_2(g)))$ and $I_{f,N}(f_s(t_4(f_s))) < I_{g,N}(g(t_4(g)))$, then let $b = s$ and go to Step 16. Otherwise continue to the next step.

Step 15. If $I_{f,N}(f_s(t_2(f_s))) < I_{g,N}(g(t_2(g)))$, then let $c = t_2(f_s)$ and go to Step 11. If $I_{f,N}(f_s(t_2(f_s))) > I_{g,N}(g(t_2(g)))$, then let $d = t_2(f_s)$ and go to Step 11. Otherwise go to Step 17.

Step 16. If $b - a < \varepsilon$, then $\log(a) \leqslant h(g) \leqslant \log(b)$. Use $\log((a + b)/2)$ to estimate $h(g)$. Otherwise go to Step 3.

Step 17. The algorithm failed.

Whenever one reaches Step 16 in the above algorithm, then one can be confident that $\log(a) \leqslant h(g) \leqslant \log(b)$, assuming that roundoff error is insignificant.

However, the above algorithm will not infallibly lead to Step 16. It could happen that for a given N and $\varepsilon > 0$ the algorithm ends in Step 17. We list the possible reasons.

1. In Step 2, the slope s could be such that $PL(s)$ contains a linear Markov map.

In this case we cannot apply Proposition 2.2 to guarantee that for some $f_t \in PL(s)$ there is a comparison $K(f_t) \geqslant K(g)$ or $K(f_t) \leqslant K(g)$, much less that $I_{f,N}(f_s(t_2(f_s))) < I_{g,N}(g(t_2(g)))$ and $I_{f,N}(f_s(t_4(f_s))) > I_{g,N}(g(t_4(g)))$ or $I_{f,N}(f_s(t_2(f_s))) > I_{g,N}(g(t_2(g)))$ and $I_{f,N}(f_s(t_4(f_s))) <$

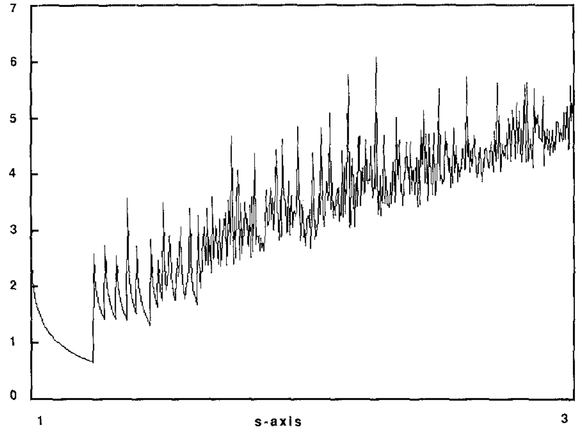

Fig. 3. Plot of the *number of digits accuracy* in estimating $s \in [1, 3]$ using the algorithm with 20 terms in the kneading matrix. Details in text.

$I_{g,N}(g(t_4(g)))$. However, there are only countably many s for which this can happen. Those s in fact are algebraic numbers and thus in theory one could choose b in Step 2 to be in the interval $(3-\varepsilon, 3)$ such that none of the slopes s that subsequently arise in Step 3 are algebraic numbers. With such a choice of starting b, if $K(g) \geqslant K(f_t)$ for some $f_t \in PL(b)$, then $b-\varepsilon \leqslant h(g) \leqslant b$ and we can estimate $h(g)$ by $b-\varepsilon/2$.

Since roundoff error is a consideration in any practical implementation of the algorithm and since the probability of situation 1 occurring is theoretically zero, we simply ignore this possibility and take our chances.

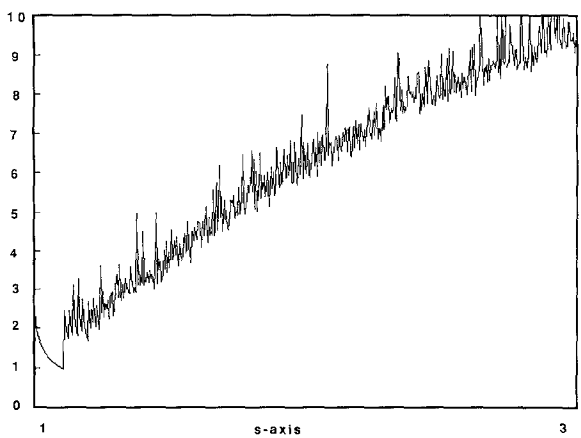

Fig. 4. Plot of the *number of digits accuracy* in estimating $s \in [1, 3]$ using the algorithm with 20 terms in the kneading matrix. Details in text.

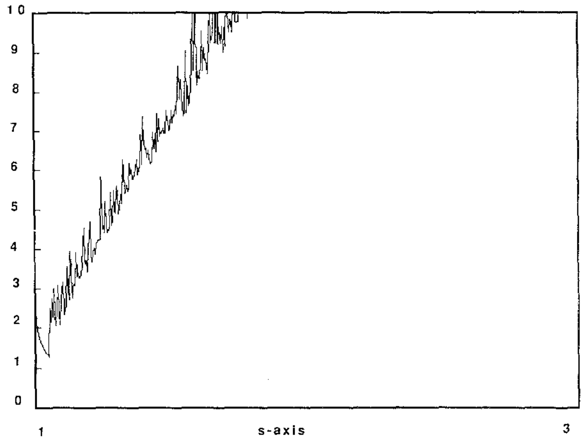

Fig. 5. Plot of the *number of digits accuracy* in estimating $s \in [1, 3]$ using the algorithm with 40 terms in the kneading matrix. Details in text.

2. For some s, t, and N, $I_{f,N}(f_s(t_2(f_s))) < I_{g,N}(g(t_2(g)))$ and $I_{f,N}(f_s(t_4(f_s))) > I_{g,N}(g(t_4(g)))$ or $I_{f,N}(f_s(t_2(f_s))) > I_{g,N}(g(t_2(g)))$ and $I_{f,N}(f_s(t_4(f_s))) < I_{g,N}(g(t_4(g)))$, but we did not apply the algorithm with sufficiently large N to obtain these comparisons.

Experience using the algorithm helps one guess if N will be sufficiently large to avoid this difficulty. However, there is a third difficulty. In theory, the algorithm could fail for all N, even if there are no linear Markov maps in $PL(s)$. We list this as follows.

3. A failure will occur if for s and t, $I_f(f_t(t_2(f_t))) = I_g(g(t_2(g)))$ or $I_f(f_t(t_4(f_t))) = I_g(g(t_4(g)))$.

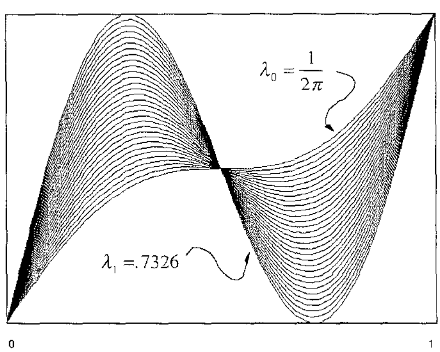

Fig. 6. The family of functions $f(x) = x + \lambda \sin(2\pi x)$ for λ in the range $\lambda_0 = 1/(2\pi)$ to $\lambda_1 = 0.7326$. The y axis is $[0, 1]$.

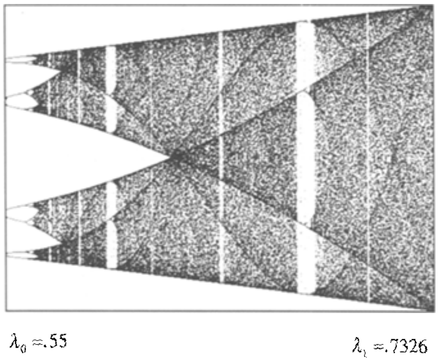

Fig. 7. Bifurcation diagram for the family of functions $f(x) = x + \lambda \sin(2\pi x)$ for λ in the range $\lambda_0 = 0.55$ to $\lambda_1 = 0.7326$. The y axis is $[0, 1]$.

In the first case $I_{f,N}(f_t(t_2(f_t))) = I_{g,N}(g(t_2(g)))$ for all N and in the second case $I_{f,N}(f_t(t_4(f_t))) = I_{g,N}(g(t_4(g)))$ for all N. Note that *both* equalities would only occur if $s = \exp(h(g))$, an occurrence having probability zero. If one of the equalities in case 3 occurs, then one would never obtain the inequalities in Steps 7, 10, or 14 for the algorithm to end in Step 16, no matter how large N might be. In practice this seems to be a rare occurrence. One can see theoretically why this is rare by observing that with an arbitrarily small change in the value of $t = t_2(f_t)$ or s the respective itineraries will no longer be equal. The next section gives a practical modification of the algorithm to minimize these difficulties. We also adduce empirical evidence that the practical modification gives the accuracy expected in the theory.

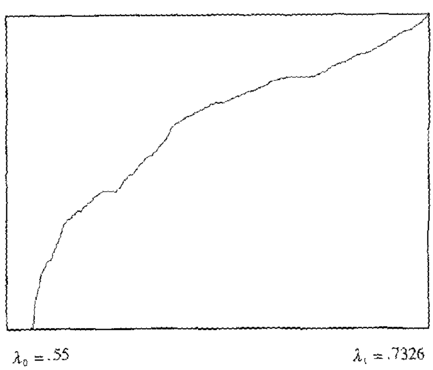

Fig. 8. Topological entropy of the family of functions $f(x) = x + \lambda \sin(2\pi x)$ for λ in the range $\lambda_0 = 0.55$ to $\lambda_1 = 0.7326$. The y axis is $[0, \log(3)]$.

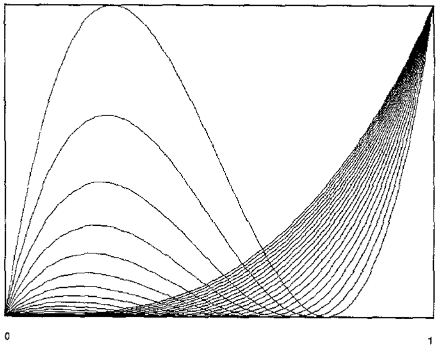

Fig. 9. Parametrized family of functions $g(x, t) = x(x - \frac{1}{2}t)^2/(1 - \frac{1}{2}t)^2$ for $0 \leqslant t \leqslant 1.5$. The x and y axes are $[0, 1]$.

6. PRACTICAL MODIFICATION OF THE ALGORITHM

Our practical implementation differs from this theoretical algorithm in an important detail; we modify Steps 7, 10, and 14 to accept equality of the respective finite itineraries as constituting a comparison.

These changes reduce the likelihood that the algorithm will fail for reason 2 or 3 at the end of Section 5. However, with this modification we no longer have any theoretical assurance of the accuracy of the numerical results. We devised an experiment to empirically test the modified algorithm. For each s with $1 < s < 3$ there is a unique $f_s \in PL(s)$ with $t_2(f_s) = 1/3$. Let $PL_{1/3}(s)$ denote the collection of these functions. For each s, $\log(s) = h(f_s)$ and in Step 16 of the original or modified algorithm $s' = (a + b)/2$ is an estimate of s. In our experiment we plot $-\log(|s' - s|)$ versus s over the interval $s \in [1, 3]$ for 534 values of s. The function

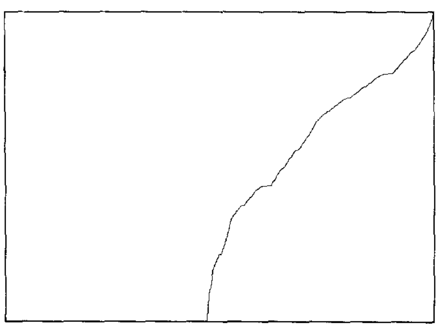

Fig. 10. Topological entropy for the functions $g(x, t) = x(x - \frac{1}{2}t)^2/(1 - \frac{1}{2}t)^2$ for $1.3 \leqslant t \leqslant 1.5$. The y axis is $[0, \log(3)]$.

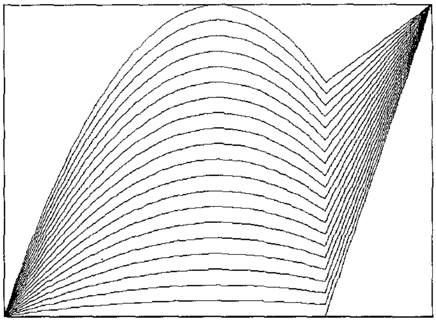

Fig. 11. Parametrized family of functions

$$f(x, t) = \begin{cases} tx(1-x) & x \leqslant 3/4 \\ (4-3t/4)(x-3/4)-3t/16 & 3/4 \leqslant x \leqslant 1 \end{cases}$$

for $0 \leqslant t \leqslant 4$. The x and y axes are $[0, 1]$.

$-\log(|s'-s|)$ gives us the *number of digits accuracy* obtained in the estimate s' by the algorithm. In Figs. 3–5 we plot the results of this experiment. In each of the three figures the bisection (Step 3 of the algorithm) was only performed 34 times. Thus, the accuracy cannot be more than $2^{-34} \approx 10^{-10}$ no matter how many terms in the kneading matrix were used. Thus there are no more than ten digits of accuracy. In Fig. 3 we used 10 terms in the kneading sequence, in Fig. 4 we used 20, and in Fig. 5 we used 40. One should compare these graphs with the graphs in Figs. 3–6 in ref. 2.

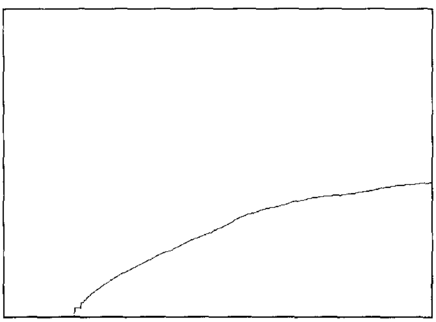

Fig. 12. Topological entropy for the functions

$$f(x, t) = \begin{cases} tx(1-x) & x \leqslant 3/4 \\ (4-3t/4)(x-3/4)-3t/16 & 3/4 \leqslant x \leqslant 1 \end{cases}$$

for $0 \leqslant t \leqslant 4$. The y axis is $[0, \log(3)]$.

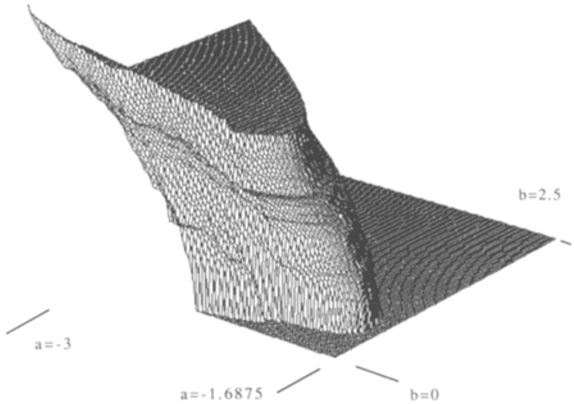

Fig. 13. The topological entropy of the parametrized family $f(x)=x^3+ax+b$ for $a \in [-3, -1.6875]$ and $b \in [0, 2.5]$. The z axis is $[0, \log(3)]$.

7. SOME EXAMPLE COMPUTATIONS

In this section we include some graphs of the topological entropy of parametrized families of functions with three monotone pieces (Figs. 6–12). The graphs were obtained using an adaptation of the algorithm given in the last section. The computation time was a few minutes for each graph using *True BASIC* on a *Macintosh IIci.*

In Fig. 13 some explanation should be given regarding our definition of the topological entropy of the maps $f_{a,b}(x)=x^3+ax+b$. Observe that f extends to a mapping f': $R \cup \{-\infty, \infty\} \to R \cup \{-\infty, \infty\}$ and that this is topologically equivalent to a mapping $g: I \to I$. We defined $h(f)$ as $h(g)$ for this associated g. There is interest in the graph of the topological entropy of this family because of a conjecture of John Milnor that the sets $E_c = \{(a, b) \mid h(f_{a,b}) = c\}$ are connected for all c. The graph lends credibility to the conjecture.

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